LOCAL GEOID MODELS

FOR

3-D HELMERT TRANSFORMATIONS

By

Henri B. Ayers

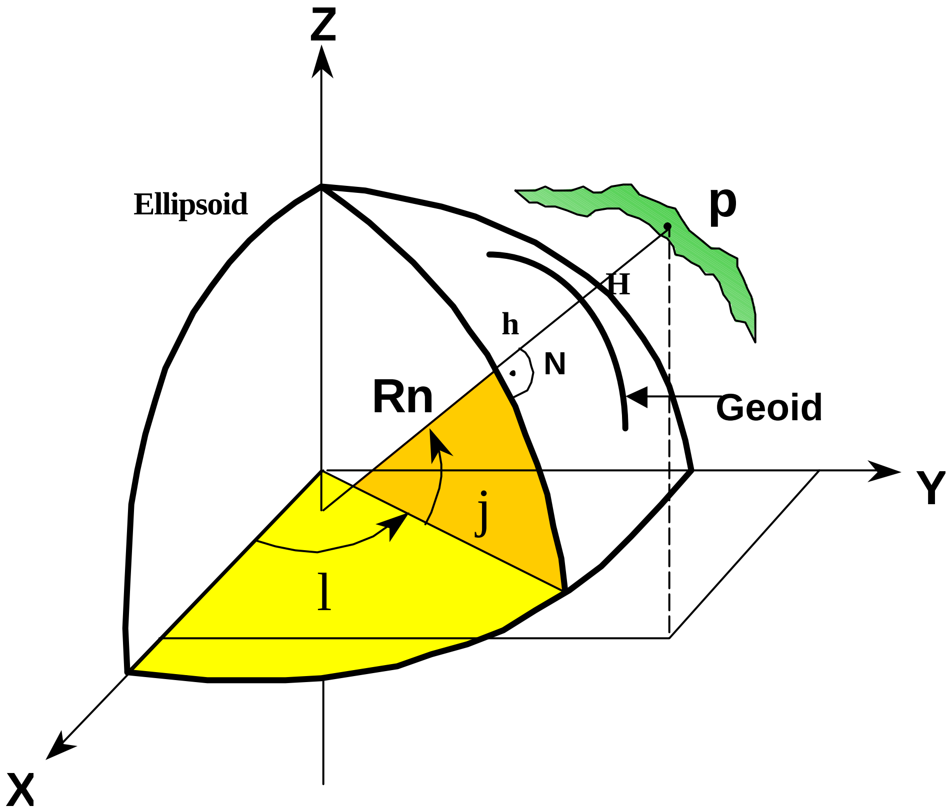
ABSTRACT

Geoid models for Local Geodetic Datums such as NAD27 and ATS77 are presented as an adequate source of ellipsoid height information to compute 3-D Helmert transformation parameters between two coordinate systems. The local Geoid models are based on the existing North American Geoid models such as Geoid99 for United States and HT1_01 for Canada. Changes in Geoid separations between points still remain the same as the ones given by these geocentric models. Local Geoid separations minimize distortions in the calculation of 3-D Helmert coordinate transformation parameters where improvements of several centimetres in height components are obtained over large areas and also over small areas under mountainous terrain conditions. Local Geoid models can also be used to convert directly post-processed or real-time GPS coordinates to Local Datum horizontal coordinates with Mean Sea Level heights using 3-D Helmert transformations.

Leica Geosystems Ltd.
May, 2000

1.0 3-D Helmert Coordinate Transformation Requirements:

One of the requirements for computing 3-D Helmert coordinate transformation parameters is to convert geodetic coordinates in 3-D cartesian units. This implies the knowledge of ellipsoid heights with respect to the ellipsoid associated to the geodetic datum. Most of the time, only orthometric heights (elevations) are known with local geodetic latitudes and longitudes. Therefore, geoid separations with respect to the local ellipsoid must be known to convert elevations to ellipsoid heights. Unfortunately, geoid separations with respect to the local datum such as NAD27 and ATS77 are not accurately known as compared with the ones related to geocentric geodetic datums such as WGS84 and NAD83. Figure 1 illustrates the relationships between 3-D cartesian coordinates with respect to geodetic coordinates, orthometric height and Geoid separation.

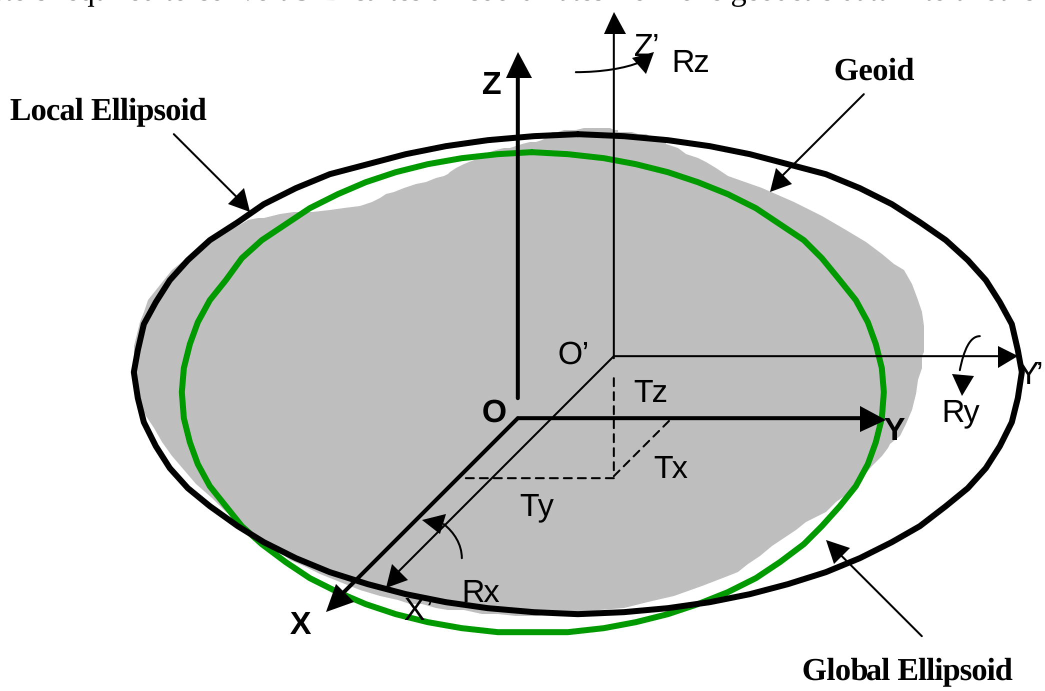


$X = (Rn + h) \cos(\varphi) \cos(\lambda)$
$Y = (Rn + h) \cos(\varphi) \sin(\lambda)$
$Z = (Rn(1 - e^2) + h) \sin(\varphi)$
$h = H + N$

where φ : Geodetic Latitude
λ : Geodetic Longitude
h : Ellipsoid Height
Rn : Prime Vertical Radius of Curvature
e^2 : First Eccentricity square
H : Orthometric Height (Elevation)
N : Geoid Separation

Figure 1: 3-D Cartesian Coordinates (X,Y,Z) from Geodetic Coordinates (φ, λ, h)

The 3-D Helmert coordinate transformation calculations involve 2 Geodetic Systems having their respective ellipsoid, origin and axis orientation. The 3-D Helmert transformation parameters consist of 3 translation components, 3 axis rotations and 1 scale factor all determined in 3-D cartesian space. Figure 2 illustrates the 3-D Helmert transformation parameters required to convert 3-D cartesian coordinates from one geodetic datum to another.



$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = Sc. \begin{bmatrix} \textbf{Rotation Matrix} \\ \textbf{(fully populated)} \end{bmatrix} . \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} Tx \\ Ty \\ Tz \end{bmatrix}$$

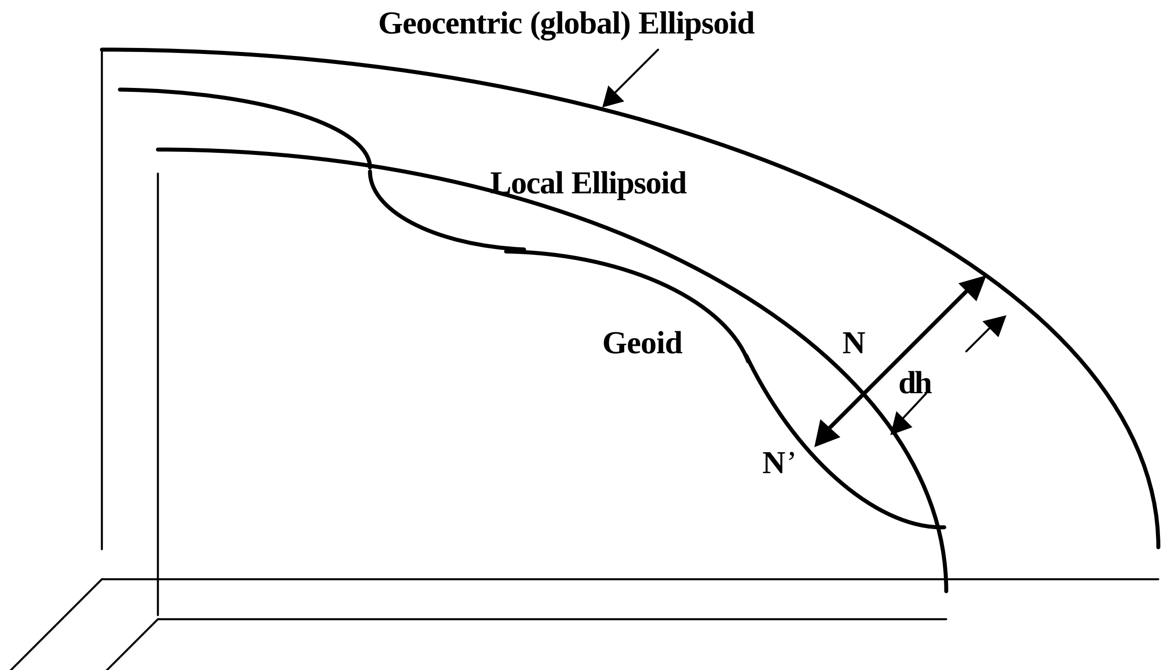
where: Tx, Ty & Tz : Translation components between the 2 origins
Rx, Ry & Rz : Rotation angles around the X, Y & Z axes
Sc : Scale Factor between the 2 coordinate systems

Figure 2: 3-D Helmert Coordinate Transformation Relationships

Omitting Geoid separations in ellipsoid heights affects all 3-D cartesian coordinate components as well as all 3-D Helmert transformation parameters. Therefore, distortions in transformed coordinates will not only appear in vertical but also in horizontal. The use of local Geoid models should provide adequate 3-D cartesian coordinates on the local geodetic system, follow the Geoid separation changes and minimize distortions in transformed coordinates.

2. Local Geoid Model Implementation:

The implementation of local Geoid models for North America consists of modifying the existing geocentric Geoid models by shifting the Geoid separations based on the change in height between the NAD83 (WGS84) and the local datums. The local latitudes and longitudes are first shifted to their corresponding geocentric values in order to obtain the geocentric geoid separations prior to shifting them to the local datum. The average datum shifts between NAD83 and local datums are: –10.0 m, 160.0 m and 180.0 m in X, Y and Z respectively for NAD27 and -4.5 m in Z for ATS77. Differences between the GRS-80 and the local (Clarke 1866 and ATS77) ellipsoids are also used to compute non-linear shifts in the height components. Figure 3 illustrates the process of shifting the geocentric Geoid separations to the local datum using Molodensky abridged formula.



$N' = N - dh$

where : N' = Local Geoid separation

N = Global Geoid separation from geocentric model

dh = Height shift between the 2 datums

$= Tx \cos(\varphi) \cos(\lambda) + Ty \cos(\varphi) \sin(\lambda) + Tz \sin(\varphi) + (a.df + f.da) \sin(\varphi) \sin(\varphi) - da$

Tx,Ty,Tz = Average translation components between the 2 datums

da = Difference between the 2 ellipsoid semi-major axes

df = Difference between the 2 ellipsoid flattenings

a = Semi-major axis of the local datum ellipsoid

f = Flattening of the local datum ellipsoid

φ = Local datum latitude, λ = Local datum longitude

Figure 3: Local Geoid Separation Calculation

Local datum Geoid separations depend on the location of points on the local ellipsoid. The Molodensky formula provides non-linear shifts in heights between the two datums. The absolute values of local Geoid separations derived from the average datum shifts between the 2 geodetic systems are only accurate within a few metres. However, relative Geoid separations between points have the same consistency as the ones derived from the Global Geoid Models.

3.0 Application of Local Geoid Models in 3-D Helmert Transformations:

The following section examines improvements in the calculations of 3-D Helmert transformation parameters when local Geoid separations are used with Orthometric heights in order to derive local ellipsoid heights.

3.1 Large Area under Flat Terrain Conditions:

One of the main features of the 3-D Helmert coordinate transformation is the ability to convert coordinates over large areas without significantly degrading the calculations. This feature is provided by the fact that the earth's curvature and meridian convergence between points are maintained by the use of the two ellipsoids, system origins in space and axis orientations. The following example illustrates the difference between 3-D Helmert calculations with and without local Geoid separations over a portion of highway of about 100 km. Points are mostly distributed in an East and West corridor where Geoid separations change by about a few metres over the area. Figure 4 shows the distribution of points over the area whereas Table 1 lists the ATS77 coordinates. Table 2 shows the comparison between 3-D Helmert calculations with Orthometric and Ellipsoid heights.

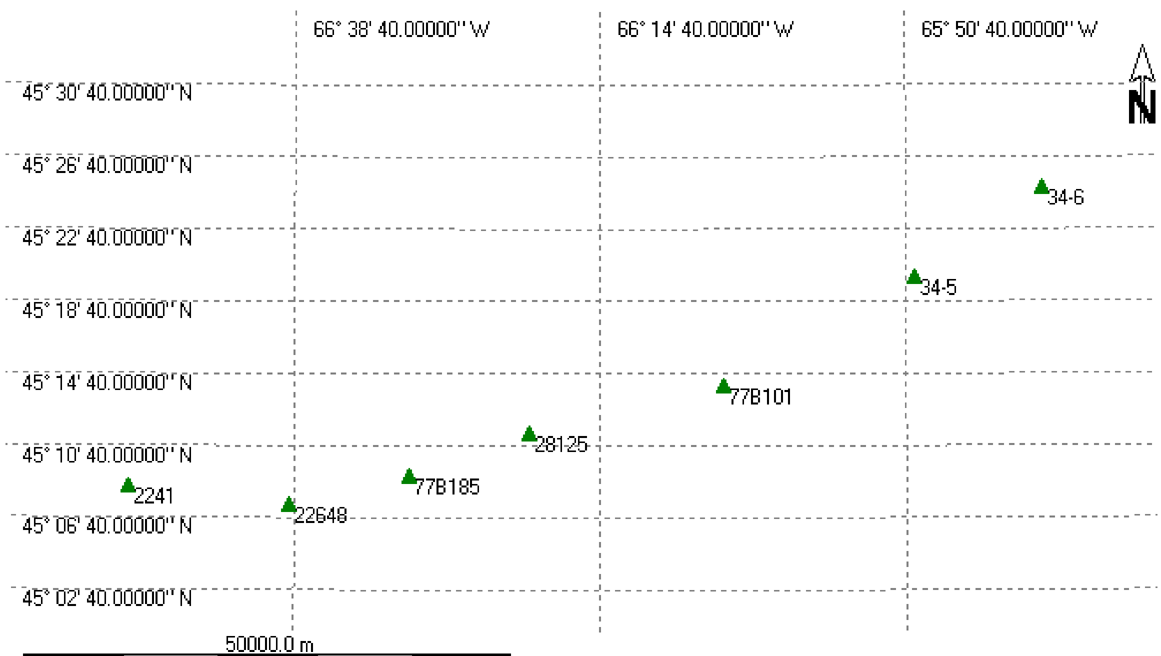


Figure 4: Distribution of Points over a Large Area under Flat Terrain Conditions

PT-ID	Latitude	Longitude	Ell. Hgt.	Orth. Hgt.	Geoid Separ.
2241	45° 08' 25.36383" N	66° 51' 42.06711" W	62.783	86.848	-24.065
22648	45° 07' 23.29809" N	66° 39' 04.01695" W	34.685	58.573	-23.888
28125	45° 11' 20.90041" N	66° 20' 16.35124" W	17.009	40.375	-23.366
34-5	45° 19' 59.90704" N	65° 50' 00.13505" W	122.298	144.435	-22.137
34-6	45° 24' 59.90405" N	65° 40' 00.13039" W	151.202	172.919	-21.717
77B101	45° 13' 59.90586" N	66° 05' 00.13259" W	25.014	47.822	-22.808
77B185	45° 08' 59.94147" N	66° 29' 38.70370" W	-3.804	19.891	-23.695

Table 1: ATS77 Geodetic Coordinates with Local Geoid Separations

System A: HIGHWAY POINTS
Ellipsoid: WGS 1984

System B: LARGE AREA-O-
Ellipsoid: ATS77
Height mode: Orthometric
Geoid Model: none

System B: LARGE AREA-E-
Ellipsoid: ATS77
Height mode: Ellipsoid
Geoid Model: GEO98-77

No.	Transformation Parameters:	Value	r.m.s.	Value	r.m.s.
1	Shift dX (m)	4.0487	0.032	-2.4925	0.0302
2	Shift dY (m)	-15.7754	0.032	-0.8808	0.0302
3	Shift dZ (m)	12.0916	0.032	-4.3035	0.0302
4	Rotation about X (")	7.2619	1.4371	-0.0728	1.358
5	Rotation about Y (")	1.0996	1.106	0.1882	1.0451
6	Rotation about Z (")	-2.1773	0.4654	-0.0818	0.4398
7	Scale (ppm)	-3.3691	0.9514	-6.9359	0.899

Residuals (m):

System A	System B	dE	dN	dH	dE	dN	dH
2241	2241	-0.1126	-0.015	0.0215	-0.1166	-0.0135	-0.0009
22648	22648	0.0173	0.0958	0.0306	0.0165	0.0951	0.0011
28125	28125	-0.0198	-0.101	-0.0534	-0.0167	-0.1016	0.0007
34-5	34-5	0.0534	-0.0388	0.0564	0.0534	-0.0394	0.0057
34-6	34-6	-0.1472	0.0124	-0.0415	-0.1512	0.0123	-0.0038
77B101	77B101	0.1427	0.0404	0.0427	0.1509	0.0408	-0.0035
77B185	77B185	0.0614	0.0061	-0.0563	0.0633	0.0061	0.0007

Table 2: Comparison between 3-D Helmert Transformation Calculations with Orthometric and Ellipsoid Heights over a Large Area under Flat Terrain Conditions

Residuals in the height components from 3-D Helmert Transformation calculations using ellipsoid heights based on a local Geoid model show an improvement of at least 10 times (from several centimetres to a few millimetres) compared with the residuals obtained from transformations using orthometric heights only. Residuals in the horizontal components are about one decimetre because of internal distortion in ATS77 geodetic latitudes and longitudes compared with the WGS84 coordinates obtained from GPS static surveys. Changes in Geoid separations are approximating changes in orthometric heights within a few centimetres over large areas under flat terrain conditions.

3.2 Small Area over Mountainous Terrain Conditions:

The 3-D Helmert coordinate transformations can still be used in small areas to adequately convert coordinates where changes in Geoid separations are significant. Approximating the Geoid surface as a tilted plane may not be sufficient to maintain vertical information from a GPS survey within one centimetre. The following example illustrates the differences in 3-D Helmert calculations with and without local Geoid separations over a small area of about 10 km in mountainous terrain. The common points are perfectly distributed in space. Geoid separations change by a few decimetres within the area. Figure 5, shows the distribution of the local points whereas Table 3 lists the NAD27 coordinates. Table 4 shows the comparison between 3-D Helmert calculations with Orthometric and Ellipsoid heights.

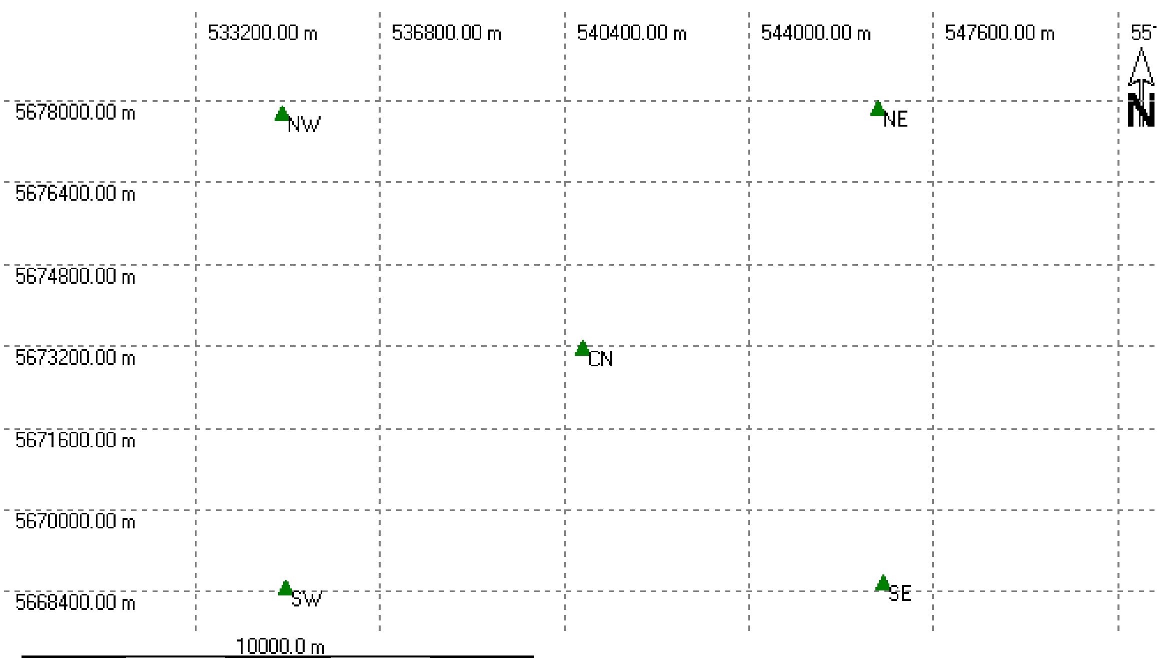


Figure 5: Distribution of Points over a Small Area in Mountainous Terrain

PT-ID	Easting (m)	Northing (m)	Ell. Hgt.	Ort. Hgt.	Geoid Sep.
CN	540825.4663	5672938.162	1971.825	1962.49	9.335
NE	546604.8701	5677620.989	2036.905	2026.991	9.914
NW	534973.4748	5677529.24	2031.76	2022.626	9.134
SE	546688.5402	5668353.831	2008.889	1999.384	9.505
SW	535034.9736	5668261.846	2021.743	2013.074	8.669

Table 3: NAD27 Geodetic Coordinates with Local Geoid Separations

System A: MOUNTAIN POINTS	System B: SMALL AREA-O	System B: SMALL AREA-E
Ellipsoid: WGS 1984	Ellipsoid: Clarke 1866	Ellipsoid: Clarke 1866
	Height mode: Orthometric	Height mode: Ellipsoid
	Geoid Model: none	Geoid Model: GEO99-27

No.	Transformation Parameters	Value	r.m.s.	Value	r.m.s
1	Shift dX (m)	5.0915	0.1147	2.4966	0.1145
2	Shift dY (m)	-159.7795	0.1147	-165.0039	0.1145
3	Shift dZ (m)	-193.2952	0.1147	-186.0375	0.1145
4	Rotation about X (")	-0.39	5.4366	-4.1359	5.4266
5	Rotation about Y (")	-21.8951	4.5212	-7.5939	4.5129
6	Rotation about Z (")	1.9929	3.9717	10.9655	3.9644
7	Scale (ppm)	-3.3691	0.9514	-6.9359	0.899

Residuals (m):

System A	System B	dE	dN	dH	dN	dE	dH
CN	CN	0.0121	-0.0033	0.0227	0.0151	-0.0013	-0.0011
NE	NE	0.1119	-0.0747	-0.0187	0.1105	-0.0764	0.0009
NW	NW	-0.4488	0.1022	0.0082	-0.4496	0.1021	0.0005
SE	SE	-0.1387	-0.1867	0.0073	-0.1384	-0.1866	-0.0004
SW	SW	0.4629	0.1625	-0.0194	0.4617	0.1622	0.0002

Table 4: Comparison between 3-D Helmert Transformation Calculations with Orthometric and Ellipsoid Heights over a Small Area under Mountainous Terrain Conditions

Residuals in the height components from 3-D Helmert Transformation calculations using ellipsoid heights based on a local Geoid model show an improvement of about 10 times (from a few centimetres to a few millimetres) compared with the residuals obtained from transformations using orthometric heights only. Residuals in the horizontal components are about a few decimetres because of high distortion in NAD27 geodetic latitudes and longitudes compared with the WGS84 coordinates obtained from GPS static surveys. Changes in Geoid separations are approximating changes in orthometric heights within one centimetre over small areas under mountainous terrain conditions.

4.0 Summary and Conclusion:

Geoid separations for local geodetic datums (NAD27 and ATS77) can be approximated by shifting Geoid separations obtained from geocentric Geoid models (Geoid99 and HT1_01). These shifts are derived from the Molodensky formula using average 3-D cartesian translation components between the geocentric and local datums and also differences in the semi-major axis and flattening between the geocentric and local ellipsoids. Changes in Geoid separations on the Local Datum still follow the relative Geoid undulations between points. Local Geoid separations minimize distortions in the calculation of 3-D Helmert coordinate transformation parameters where improvements of several centimetres in transformed coordinates are obtained over large areas and also over small areas under mountainous terrain conditions. These improvements are indicated in Tables 2 and 4 by examining residuals in the height components from 3-D Helmert transformations computed with and without local Geoid separations.

Local Geoid models can then be used to convert directly post-processed or real-time GPS coordinates to Local Datum horizontal coordinates with Mean Sea Level heights using 3-D Helmert transformations. Local Geoid separations provide better height transformations not only over large areas (10 km or more) but also over small areas (10 km or less) under mountainous terrain conditions. A best-fit plane may not always keep vertical transformations under one centimetre because of non-linear changes in Geoid separations between points.

5.0 References:

Krakiwsky E.J. and Wells, D.E. (1971) “ Coordinate Systems in Geodesy “, University of New Brunswick, Department of Geomatics Engineering and Geodesy, Lecture Notes # 16

Leick, A. (1990) . “ GPS Satellite Surveying “, First Edition, Wiley-Interscience Publication, ISBN 0-471-81990-5

Hoffmann-Wellenhof, B., Lichtenegger, H., and Collins, J. (1992) “ GPS Theory and Practice “, Springer-Verlag Publication, ISBN 3-211-82477-4